A vexing math problem finds an elegant solution

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(Phys.org) —A famous math problem that has vexed mathematicians for decades has met an elegant solution by Cornell researchers. Graduate student Yash Lodha, working with Justin Moore, professor of mathematics, has described a geometric solution for the von Neumann-Day problem, first described by mathematician John von Neumann in 1929.

Lodha presented his solution at the London Mathematical Society's Geometric and Cohomological Group Theory symposium in August, and has submitted the work to a journal. "People were very excited by this," Lodha said. "[The solution] is natural and compelling enough to study for its own sake."

Lodha works in the field of geometric group theory. A group is a mathematical construct that describes the notion of the symmetries of an object, whether it's a physical object or a theoretical space. For example, a polygon has rotational as well as reflectional symmetries, all of which, together with the operation of composition, form what's called a finite group, because the polygon can be described as a finite sequence of operations that reflect its symmetries.

Formally, a group can be described as words in an alphabet together with a set of rules that are called "relations." Group theorists, Lodha said, are like biologists who classify species; mathematicians try to categorize groups that have properties A, B or C – but is there one that has A but not C?

The inspiration for Lodha's work originated in the early 20th century, when mathematicians first proved that a ball that exists in three-dimensional space can be chopped into a finite number of pieces – "like tearing up a piece of paper without stretching or squeezing," Lodha explained – and can be reassembled, like a jigsaw puzzle, into two balls, each the size of the original ball. This is known as the Banach-Tarski paradox.

von Neumann, in studying this paradox, was the first to describe the reason behind it: He attributed it not to the geometry of 3-D space, but to the algebraic properties of the symmetries inherent to the sphere. He was the first to isolate this property, which mathematicians today call "non-amenability."

von Neumann further observed that if a group contains free groups, which are groups that have a finite alphabet and no rules, then it must be non-amenable. He posed the question of whether the opposite is true – are there groups that do not contain free groups and are also non-amenable? The problem, later popularized by M.M. Day, waited another 40 years before mathematician Alexander Olshanskii cracked it, although Olshanskii's group had an infinite set of rules.

Another two decades went by before Olshanskii and Mark Sapir supplied another solution in response to the von Neumann-Day problem. This time, their example was governed by a finite, but astronomically large set of rules – close to 10200. It also lacked a natural geometric model. So mathematicians probed further for a group with a finite set of rules, that is non-amenable and does not contain free groups.

For the first time, Lodha describes a group that has only nine rules, a natural geometric model, is non-amenable and does not contain free groups.

Advances in mathematics are almost always incremental and build upon previous work, Lodha said. To complete this work, among his most valuable insights was one first described by the late Bill Thurston, Fields medalist and Cornell's Jacob Gould Schurman Professor of Mathematics, which involved a way of expressing the group in a different light, as a "continued fractions model."

Lodha's work also builds heavily on work by Nicolas Monod, who constructed a geometrically oriented, but not finitely presented, counterexample to the von Neumann-Day problem. Lodha and Moore's contribution was to isolate a finitely presented subgroup, with only nine relations, of Monod's example.

Further work on the group, which doesn't yet have a name, could make the solution to the von Neumann-Day problem even stronger: by isolating stronger finiteness conditions for proving that the group has a finite number of rules.

More information: xxx.tau.ac.il/abs/1308.4250v1

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