Mathematicians Solve 140-Year-Old Boltzmann Equation

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(PhysOrg.com) -- Two University of Pennsylvania mathematicians have found solutions to a 140-year-old, 7-dimensional equation that were not known to exist for more than a century despite its widespread use in modeling the behavior of gases.

The study, part historical journey but mostly mathematical proof, was conducted by Philip T. Gressman and Robert M. Strain of Penn's Department of Mathematics. The solution of the Boltzmann equation problem was published in the Proceedings of the National Academy of Sciences. Solutions of this equation, beyond current computational capabilities, describe the location of gas molecules probabilistically and predict the likelihood that a molecule will reside at any particular location and have a particular momentum at any given time in the future.

During the late 1860s and 1870s, physicists James Clerk Maxwell and Ludwig Boltzmann developed this equation to predict how gaseous material distributes itself in space and how it responds to changes in things like temperature, pressure or velocity.

The equation maintains a significant place in history because it modeled gaseous behavior well, and the predictions it led to were backed up by experimentation. Despite its notable leap of faith -- the assumption that gases are made of molecules, a theory yet to achieve public acceptance at the time — it was fully adopted. It provided important predictions, the most fundamental and intuitively natural of which was that gasses naturally settle to an equilibrium state when they are not subject to any

sort of external influence. One of the most important physical insights of the equation is that even when a gas appears to be macroscopically at rest, there is a frenzy of molecular activity in the form of collisions. While these collisions cannot be observed, they account for gas temperature.

Gressman and Strain were intrigued by this mysterious equation that illustrated the behavior of the physical world, yet for which its discoverers could only find solutions for gasses in perfect equilibrium.

Using modern mathematical techniques from the fields of partial differential equations and harmonic analysis — many of which were developed during the last five to 50 years, and thus relatively new to mathematics — the Penn mathematicians proved the global existence of classical solutions and rapid time decay to equilibrium for the Boltzmann equation with long-range interactions. Global existence and rapid decay imply that the equation correctly predicts that the solutions will continue to fit the system's behavior and not undergo any mathematical catastrophes such as a breakdown of the equation's integrity caused by a minor change within the equation. Rapid decay to equilibrium means that the effect of an initial small disturbance in the gas is short-lived and quickly becomes unnoticeable.

"Even if one assumes that the equation has solutions, it is possible that the solutions lead to a catastrophe, like how it's theoretically possible to balance a needle on its tip, but in practice even infinitesimal imperfections cause it to fall over," Gressman said.

The study also provides a new understanding of the effects due to grazing collisions, when neighboring molecules just glance off one another rather than collide head on. These glancing collisions turn out to be dominant type of collision for the full Boltzmann equation with long-range interactions.

“We consider it remarkable that this equation, derived by Boltzmann and Maxwell in 1867 and 1872, grants a fundamental example where a range of geometric fractional derivatives occur in a physical model of the natural world,” Strain said. “The mathematical techniques needed to study such phenomena were only developed in the modern era.”

Provided by University of Pennsylvania