New mathematical framework pioneers a theory of stability of interacting systems

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Mathematical concepts of stability and instability also apply to “dynamic” processes, such as the beating of the heart and other biological oscillatory processes. Credit: Lancaster University

Researchers have introduced a new mathematical framework for describing the stability of processes in a landmark paper.

The framework is designed to be applicable to scenarios in which the process is influenced by external factors that vary relatively slowly over time.

The work by researchers from Lancaster University and the University of Exeter is published in the journal *Chaos*.

The authors—Dr. Julian Newman, Dr. Maxime Lucas and Professor Aneta Stefanovska—have initiated the development of a new approach to mathematically defining and quantifying stability, applicable to finite-time processes subject to reasonably gradual changes in the external environment.

Professor Aneta Stefanovska, of the Nonlinear and Biomedical Physics group at Lancaster University, has had a long-standing passion for the development of both theoretical and experimental analysis tools for studying systems that interact with their environment, such as biological processes.

Dr. Lucas from Lancaster University said: "Suppose you have a ball and put it down at the bottom of a bowl. If you give it a little nudge, the ball will return to its original position at the bottom of the bowl. We say that the ball is in a 'stable' position because it recovers from being given a little nudge. However, anywhere on the edge of the bowl, it will be in an 'unstable' position: the slightest nudge will make it move far away from its original position."

In the late 19th century, the Russian mathematician Aleksandr Lyapunov introduced precise mathematical definitions of these concepts of stability and instability, as well as of how to quantify the level of stability or instability.

These concepts apply to static things like a ball, but also apply to more "dynamic" processes, such as the beating of the heart and other biological oscillatory processes. In fact, our heart rate is stable because it always stays within normal range despite being continuously pushed higher or lower by the ever-changing needs of the body.

The mathematical framework introduced by Lyapunov has dominated stability analysis in the sciences for many decades. It was designed for processes that evolve independently of their environment. As a corollary, it assumes the process follows a "law" that describes its behavior indefinitely into the future. This is very well suited to describe processes such as Newton's "law of gravity."

Dr. Newman from the University of Exeter said: "Many processes, however, interact heavily with their environment; we call them 'thermodynamically open.' In fact, 'biological death' may be defined as the loss of the ability of an organism's internal biological processes to do so. Because of this, Lyapunov's mathematical stability definitions could potentially be inapplicable to the study of, say, a biological oscillator or a subsystem of the Earth's climate. The problem lies not in the work of Aleksandr Lyapunov himself, whose work was an absolute milestone in the history of science, but rather in the issue of where and how to appropriately apply his work. Our work lays the foundation for a new stability theory applicable to situations where, in the absence of such new theory, one might inappropriately try to apply traditional analysis tools based on Lyapunov's framework and come to incorrect conclusions."

Lyapunov's approach relies on a kind of "infinite-time approximation," and the potential inapplicability of such an approach to the study of thermodynamically open processes has, in recent decades, led to a slowly growing interest in the topic of mathematical analysis of "explicitly finite-time" dynamical processes.

The authors' new approach to mathematically defining and quantifying stability reframes Lyapunov's "infinite-time approximation" definitions in terms of a very different approximation, namely the so-called "slow-fast" setup where sufficiently gradual changes are approximated by "infinitely steady" changes.

The paper gives mathematical definitions and results only for the "simplest kind of process" (namely oscillatory processes whose behavior can be approximated by a model that only tracks the progress of a single variable); however, the authors expect that the new approach initiated in this paper will open up a whole new branch of the mathematical disciplines of stability theory and bifurcation theory, and applications in practically every aspect of science.

More information: J. Newman et al, Stabilization of cyclic processes by slowly varying forcing, *Chaos: An Interdisciplinary Journal of Nonlinear Science* (2021). DOI: 10.1063/5.0066641

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