Mathematician untangles legendary problem

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Karl Mahlburg, a young mathematician, has solved a crucial chunk of a puzzle that has haunted number theorists since the math legend Srinivasa Ramanujan scribbled his revolutionary notions into a tattered notebook. "In a nutshell, this [work] is the final chapter in one of the most famous subjects in the story of Ramanujan," says Ken Ono, Mahlburg's graduate advisor and an expert on Ramanujan's work. Ono is a Manasse Professor of Letters and Science in mathematics.

"Mahlburg's achievement is a striking one, " agrees George Andrews, a mathematics professor at Penn State University who has also worked deeply with Ramanujan's ideas.

The father of modern number theory, Ramanujan died prematurely in 1920 at the age of 32. The Indian mathematician's work is vast but he is particularly famous for noticing curious patterns in the way whole numbers can be broken down into sums of smaller numbers, or "partitions." The number 4, for example, has five partitions because it can be expressed in five ways, including 4, 3+1, 2+2, 1+1+2, and 1+1+1+1.

Ramanujan, who had little formal training in mathematics, made partition lists for the first 200 integers and observed a peculiar regularity. For any number that ends in 4 or 9, he found, the number of partitions is always divisible by 5. Similarly, starting at 5, the number of partitions for every seventh integer is a multiple of 7, and, starting with 6, the partitions for every 11th integer are a multiple of 11.

The finding was an intriguing one, says Richard Askey a emeritus mathematics professor who also works with aspects of Ramanujan's work. "There was no reason at all that multiplicative behaviors should have anything to do with additive structures involved in partitions."

The strange numerical relationships Ramanujan discovered, now called the three Ramanujan "congruences," mystified scores of number theorists. During the Second World War, one mathematician and physicist named Freeman Dyson began to search for more elementary ways to prove Ramanujan's congruences. He developed a tool, called a "rank," that allowed him to split partitions of whole numbers into numerical groups of equal sizes. The idea worked with 5 and 7 but did not extend to 11. Dyson postulated that there must be a mathematical tool--what he jokingly called a "crank"--that could apply to all three congruences.

Four decades later, Andrews and fellow mathematician Frank Garvan discovered the elusive crank function and for the moment, at least, the congruence chapter seemed complete.

But in a chance turn of events in the late nineties, Ono came upon one of Ramanujan's original notebooks. Looking through the illegible scrawl, he noticed an obscure numerical formula that seemed to have no connection to partitions, but was strangely associated with unrelated work Ono was doing at the time.

"I was floored," recalls Ono.

Following the lead, Ono quickly made the startling discovery that partition congruences not only exist for the prime number 5, 7 and 11, but can be found for all larger primes. To prove this, Ono found a connection between partition numbers and special mathematical relationships called modular forms.

But now that Ono had unveiled infinite numbers of partition congruences, the obvious question was whether the crank universally applied to all of them. In what Ono calls "a fantastically clever argument," Mahlburg has shown that it does.

A UW-Madison doctoral student, Mahlburg says he spent a year manipulating "ugly, horribly complicated" numerical formulae, or functions, that emerged when he applied the crank tool to various prime numbers. "Though I was working with a large collection of functions, under the surface I slowly began to see a uniformity between them," says Mahlburg.

Building on Ono's work with modular forms, Mahlburg found that instead of dividing numbers into equal groups, such as putting the number 115 into five equal groups of 23 (which are not multiples of 5), the partition congruence idea still holds if numbers are broken down differently. In other words, 115 could also break down as 25, 25, 25, 10 and 30. Since each part is a multiple of 5, it follows that the sum of the parts is also a multiple of 5. Mahlburg shows the idea extends to every prime number.

"This is an incredible result," says Askey.

Mahlburg's work completes the hunt for the crank function, says Penn State's Andrews, but is only a "tidy beginning" to the quest for simpler proofs of Ramanujan's findings. "Mahlburg has shown the great depth of one particular well that Ramanujan drew interesting things out of," Andrews adds, "but there are still plenty of wells we don't understand."

Source: University of Wisconsin-Madison